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On group rings with restricted minimum condition

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Abstract

In this paper we investigate the group rings RG satisfying the restricted minimum condition.

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1. Results

Let R be an associative ring with unit element. R is said to satisfy the left restricted minimum condition, if for each nontrivial ideal J of R the ring R/J is left artinian. In this paper we consider the group rings with left restricted minimum condition, in the case when RG itself is not left artinian.

We prove the following:

Theorem 1.1. *Let G be a group with non-trivial center and let R be a commutative ring with unit element. If the group ring RG satisfies the left restricted minimum condition, then R is left artinian and either G is finite, or G is the infinite cyclic group.*

For group algebras the converse assertion is also true.

Theorem 1.2. *Let G be a group with non-trivial center and let R be a field. The group algebra RG satisfies the left restricted minimum condition if and only if either G is finite, or G is the infinite cyclic group.*

By $A(RG)$ we mean the augmentation ideal of RG, that is the kernel of the ring homomorphism $\phi : RG \to R$ sending each group element to 1. It is easy to see that

$A(RG)$ is a free R-module in which the set of the elements $g-1$ with $1 \neq g \in G$ form a basis. For a normal subgroup H of G we denote by $I(H)$ the ideal of RG generated by all elements of the form $h-1$ with $h \in H$. As it is well-known, $I(H)$ is the kernel of the natural epimorphism $\overline{\phi} : RG \to R[G/H]$ induced by the group homomorphism ϕ of G onto G/H, furthermore

$$RG/I(H) \cong R[G/H], \tag{1.1}$$

and $I(G) = A(RG)$.

The commutator subgroup and the center of the group G will be denoted by G' and $\zeta(G)$, respectively.

2. Proof of Theorems

We need the following two statements.

Proposition 2.1 (Theorem 4.12 in [2]). *If G is a group whose center has finite index n, then G' is finite and $(G')^n = 1$.*

Proposition 2.2 (Theorem 4.33 in [2]). *An infinite group has each non-trivial subgroup of finite index if and only if it is infinite cyclic.*

Proof of Theorem 1.1. It is well-known that the group ring RG is left artinian if and only if R is left artinian and G is finite. Assume that RG satisfies the left restricted minimum condition. According to (1.1) for every normal subgroup H the factor group G/H is finite and from the isomorphism $RG/A(RG) \cong R$ it follows that R is left artinian. Furthermore, $RG/I(\zeta(G))$ is left artinian and therefore, by (1.1), $G/\zeta(G)$ is finite. Then Proposition 2.1 guarantees that G' is finite. If $G' \neq 1$ then, by (1.1) G/G' is finite, and so G is finite. On the other hand, if G is abelian and infinite, then by (1.1) we have that every non-trivial subgroup of G has finite index. But then Proposition 2.2 states that G is the infinite cyclic group and the proof of the theorem is complete. □

Let R be an euclidean ring with the euclidean norm φ such that $\varphi(ab) \geqslant \varphi(a)$ for all $a \neq 0$, $b \neq 0$ $(a, b \in R.)$ Then R is a principal ideal ring. Let $I = (r)$ and $J = (s)$ be the ideals of R generated by the element r and s respectively, and assume that $I \supseteq J$. Then $s = rt$ for a suitable $t \in R$, and $\varphi(s) = \varphi(rt) \geqslant \varphi(r)$. It is easy to see that $\varphi(e) = 1$ if and only if e is an unit in R and that $I = J$ if and only if $\varphi(r) = \varphi(s)$.

Let $J = (s)$ be an arbitrary ideal of an euclidean ring R and let

$$\overline{R} \supseteq \overline{J}_1 \supseteq \overline{J}_2 \supseteq \ldots \supseteq \overline{J}_n \supseteq \ldots \supseteq \bigcap_{i=1}^{\infty} \overline{J}_i = \overline{J}_\omega \tag{2.1}$$

a sequence of ideals, where $\overline{R} = R/J$ and ω the first limit ordinal. Denote by J_k the inverse image of $\overline{J}_k$ in R $(k = 1, 2, \ldots$ or $k = \omega)$. Then J_k's are principal ideals and, in view of (2.1) we have that

$$R \supseteq J_1 \supseteq J_2 \supseteq \ldots \supseteq J_n \supseteq \ldots \supseteq J_\omega \supseteq J = (s). \tag{2.2}$$

Suppose that $J_k = (s_k)$. Since $J_k \supseteq J = (s)$, so $\varphi(s) \geqslant \varphi(s_k)$ for all k $(k = 1, 2, \ldots$ and $k = \omega)$ But $\varphi(s)$ and $\varphi(s_k)$ are non-negative integers, therefore there exists a natural number n such that $\varphi(s_n) = \varphi(s_{n+1}) = \ldots = \varphi(s)$. Thus the sequence (2.2) has finite length and consequently, the sequence (2.1) is finite, too. It follows that for each ideal J of R the ring R/J is artinian, and we have

Lemma 2.3. *Euclidean rings satisfy the restricted minimum condition.*

It was prowed in [1] that the group algebra of the infinite cyclic group over a field is an euclidean ring. Hence, Theorem 1.2 is a direct consequence of Lemma 2.3 and Theorem 1.1.

References

[1] Király, B., Orosz, Gyuláné, Egy euklideszi gyűrű, *Acad. Paed. Agriensis, Sect. Math.* (1998), 71–76.

[2] Robinson, J.S., Finiteness Conditions and Generalized Soluble Groups, Part 1, *Springer-Verlag New York Heildelberg Berlin*, 1972.

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